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The Development of CFD for Image Watermarking Problem for Security Based on Asymmetric Cryptography Using Edge Irregular Total k-Labeling of Signed Graph

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Abstract. *A signed graph is a graph where each edge is assigned a positive or negative sign. Formally, a signed graph is defined as a triplet $\Sigma = (\mathbb{G}, \sigma)$, where: $\mathbb{G} = (V, E)$ is the underlying graph, consisting of a set of vertices V and a set of edges E. Furthermore $\sigma : E \to \{+, -\}$ is a function assigning a sign (positive or negative) to each edge. An edge irregular total k-labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ of a signed graph $\mathbb{G} = (V, E)$ assigns labels to both the vertices and edges of $\mathbb{G}$ such that, for any two distinct edges uv and $u'v'$, their respective weights $f(u) + \sigma f(uv) + f(v)$ and $f(u') + \sigma f(u'v') + f(v')$ are distinct. The total edge irregularity strength $tes(\mathbb{G})$ is defined as the smallest k for which $\mathbb{G}$ admits such a labeling. In this paper, we determine the total edge irregularity strength for specific classes of signed graphs and determined the sharpest lower bound of $tes(\mathbb{G})$. Furthermore, based on these theoretical results, we propose a Conceptual Framework Design (CFD) for secure image watermarking that integrates edge irregular total k-labeling of signed graphs with asymmetric cryptography. The CFD generates public, private, and shift keys from the labeling structure to control the embedding and verification process. Both the image and labeled graph are converted to binary matrices, combined at the block level, then hashed using SHA-256 and encrypted with a private key to produce a digital signature. This approach enhances the robustness, authenticity, and security of the watermarked image.*
Keywords: Edge Irregular Total k-Labeling, Signed Graph, Asymmetric Cryptography, Image Watermarking, Conceptual Framework Design (CFD).

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1. **Introduction.** A signed graph is a mathematical structure that generalizes a traditional graph by assigning a positive or negative sign to each edge [1]. Formally, a signed graph $G = (V, E, \sigma)$ consists of a vertex set V, an edge set E, and a signature function $\sigma : E \to \{+, -\}$ [2]. This additional structure provides a richer modeling framework for complex systems in which interactions may be cooperative (positive) or antagonistic (negative) [3, 4]. Such modeling proves valuable in real-world contexts like trust and conflict dynamics in social networks, chemical bond interactions, and spin arrangements in physics.

One of the central theoretical developments in the study of signed graphs is the concept of balance, originally introduced by Harary [5]. A signed graph is said to be balanced if every cycle in the graph has an even number of negative edges, or equivalently, the product of edge signs in every cycle is positive [6]. Balanced signed graphs tend to represent stable systems [7], while unbalanced ones indicate structural tension or inconsistency. Understanding balance has been instrumental in the structural analysis of social relationships and system stability across multiple disciplines.

As signed graphs continue to attract attention, researchers have begun to explore various labeling techniques applied to these structures. Labeling in graph theory assigns values—often integers—to the elements of a graph under certain rules or constraints [8], and it plays a crucial role in measuring and controlling structural properties. One significant and emerging direction within this context is the investigation of edge irregular total k-labelings on signed graphs, which seeks to extend classical labeling concepts by incorporating the sign of edges. This naturally leads to the study of labeling-based parameters such as the total edge irregularity strength (TES), a measure of irregularity in edge-weight distributions.

The concept of TES was first introduced by Kandi et al. [9], who studied edge weight distributions in standard graphs. Further research by Bacáková et al. [10] explored TES for trees, cycles, and complete graphs, deriving tight bounds for these families. Ivančo and Jendrol' [11] focused specifically on the total edge irregularity strength of trees, presenting results that expanded the foundational understanding of TES in acyclic structures. Later, Ahmad et al. [12] extended the investigation to more complex structures, including bipartite and planar graphs. More recent work by Marić et al. [13] focused on the application of TES to edge-weighted networks, highlighting its importance in communication networks and scheduling problems.

The total edge irregularity strength of a graph $G = (V, E)$, denoted by $\mathrm{tes}(G)$, is defined as the minimum value k such that there exists a labeling $\nu : V \cup E \to \{1, 2, \dots, k\}$ for which every pair of distinct edges e and g satisfies $\varphi(e) \neq \varphi(g)$, where the weight of an edge $e = \{u, v\}$ is defined as

$$\varphi(e) = \nu(u) + \nu(v) + \nu(e).$$

The smallest such k is called the *total edge irregularity strength* of G, denoted $\mathrm{tes}(G)$. As mentioned in [14], a lower bound on $\mathrm{tes}(G)$ has been established.

This topic stems from a broader problem introduced by Chartrand et al. [15], who proposed assigning positive integer labels to the edges of a connected graph of order at least three such that the graph becomes irregular—that is, the sum of labels on edges incident to each vertex is distinct. This led to the development of a graph parameter known as the *irregularity strength*, denoted $s(G)$. Determining $s(G)$ is non-trivial even for simple structures such as paths and grids [16, 17, 18].

Several researchers, including Amar and Togni [19], Jacobson and Lehel [15], and Nierhoff [20], have studied irregularity labeling in various graph families. Karonski, Łuczak, and Thomason [17] conjectured that the edges of any connected graph of order at least three can be labeled with integers from $\{1, 2, 3\}$ such that adjacent vertices have different incident edge sums. Inspired by this, Bača et al. [10] introduced the concept of total edge irregularity strength as a natural extension into the domain of total labeling. Other studies on total edge irregularity strength include investigations on the generalized prism graph [21], the subdivision of star graphs [22], the generalized helm graph [23], as well as various graph variants such as star, double-star, and caterpillar graphs [24, 25, 26].

Despite these advancements, the study of total edge irregularity strength has been primarily limited to unsigned graphs, leaving a significant gap in the exploration of this parameter in signed graphs, where edge polarity adds an additional layer of complexity. This raises fundamental questions about how edge signs influence the minimum weight assignment and whether established bounds for TES in unsigned graphs remain valid when extended to signed graphs. This research seeks to address this gap by investigating the behavior of TES on signed graphs, deriving new bounds, and developing methods to determine the minimum weight assignment under the influence of edge signs, thus extending the theoretical and practical understanding of signed graphs.

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In addition to this theoretical focus, the study also introduces a Conceptual Framework Design (CFD) for secure image watermarking that combines edge irregular total k-labeling of signed graphs with asymmetric cryptography. The growing relevance of graph-based structures in cryptographic and watermarking systems has been explored in recent studies such as Kiswara et al. [27] and Dafik et al. [28]. Dafik et al. [29], who analyzed its application in secret sharing schemes involving the affine cipher. Related works by Sao et al. [30] and Dahshan et al. [31] further reinforce the integration of reversible data hiding and cryptographic access control mechanisms to enhance data confidentiality and integrity. Inspired by these developments, the proposed CFD framework utilizes the structure of labeled signed graphs to generate cryptographic keys, enabling watermark embedding and verification through digital signature techniques. This approach establishes a novel intersection between discrete mathematical graph theory and real-world security applications.

2. **Methods.** This research employs deductive analytic methods to determine the total edge irregularity strength (TES) of signed graphs, incorporating the additional constraint that the edge weights must form an arithmetic sequence. The procedure begins by defining the signed graph $\mathbb{G} = (V, E, \sigma)$, where V is the set of vertices, E is the set of edges, and $\sigma : E \to \{+, -\}$ assigns a sign to each edge. A critical step involves constructing a weight assignment $f : E \to \{a, a+d, a+2d, \dots\}$, where a and d are integers, and ensuring that f satisfies the conditions for TES.

The method starts by identifying the upper bound of $\text{tes}(\mathbb{G})$, which provides insight into the minimum k required for a valid edge labeling. The structure of $\mathbb{G}$ is then analyzed to determine its order (number of vertices $|V|$) and size (number of edges $|E|$), as well as the distribution of positive and negative edges as defined by σ. Edges with *positive signs* are represented by solid lines, while edges with *negative signs* are represented by dashed lines. For dashed lines, the weight function $f(e)$ is considered negative during calculations. This distinction ensures that the polarity of the edge is explicitly accounted for in the total weight computation.

A bijective weight function f is constructed under the arithmetic constraint, ensuring that the total edge weights $w(uv)$, computed from the labels of the incident vertices and the edge itself, are distinct for all edges $uv \in E(G)$. Specifically, for each edge $uv \in E(G)$, the edge weight is defined as:

$$w(uv) = f(u) + f(v) + \sigma(uv) \cdot f(uv),$$

where $\sigma(uv) = +1$ for solid lines (positive edges) and $\sigma(uv) = -1$ for dashed lines (negative edges). This ensures that negative edges contribute negatively to the total edge weight, reflecting the polarity imposed by σ. The condition $w(uv) \neq w(xy)$ for all $uv, xy \in E$, with $uv \neq xy$, is verified to satisfy the edge irregularity requirement of the TES parameter.

Upon determining a valid labeling, the edge weight function w is analyzed to evaluate its adherence to both the signed graph structure and the arithmetic constraint. This step is crucial, as it demonstrates the existence of a labeling that minimizes k while satisfying both irregularity and arithmetic requirements. Finally, with the obtained TES value for $\mathbb{G}$, the results are analyzed for specific families of signed graphs, and new theoretical bounds or properties are derived. This contributes to a deeper understanding of the interplay between edge polarity and weight constraints in signed graph labeling theory.

3. **Research Findings.** In this section, the research findings are divided into two main parts. The first part focuses on the theoretical results related to the lower bound of total edge strength in signed graphs, as well as the calculation of total edge strength for several specific classes of signed graphs. These mathematical results serve as the foundational support for the labeling method used in this study. The second part elaborates on the development of the Conceptual Framework Design (CFD), which integrates asymmetric cryptography and edge irregular total k-labeling of signed graphs to construct a secure and robust image watermarking system.

3.1. **Theoretical Results on Edge Strength.** In this section, we will show the lower bound of the total edge strength of signed graph and the total edge strength of some signed graphs.

Theorem 1. *If* $\mathbb{G}$ *is signed graph, then*

$$tes(\mathbb{G}) \geq \left\lceil \frac{|E(\mathbb{G})| + 1}{4} \right\rceil$$

Proof. Let k be the largest label of the total edge irregularity strength. The first step is to determine the minimum and maximum bounds of $W(E^+)$, which represents the total weight assigned to the edges of the graph $\mathbb{G}$. The minimum weight of $W(E^+)$, $W(E^+)_{\min}$, is obtained when all edge weights take the smallest values in the arithmetic sequence. Thus:

$$W(E^+)_{\min} = 1 + 1 + 1 = 3.$$

On the other hand, the maximum weight of $W(E^+)$, $W(E^+)_{\max}$, occurs when all edge weights are at their maximum values in the sequence. Hence:

$$W(E^+)_{\max} = k + k + k = 3k.$$

Therefore, the range of $W(E^+)$ can be expressed as:

$$3 \leq W(E^+) \leq 3k.$$

The next step is to derive the bounds for $W(E^-)$. The minimum weight of $W(E^-)$, $W(E^-)_{\min}$, is obtained when all edge weights take the smallest values in the arithmetic sequence. Thus:

$$W(E^-)_{\min} = 1 - k + 1 = 2 - k.$$

On the other hand, the maximum weight of $W(E^-)$, $W(E^-)_{\max}$, occurs when all edge weights are at their maximum values in the sequence. Hence:

$$W(E^+)_{\max} = k - 1 + k = 2k - 1.$$

Therefore, the range of $W(E^-)$ can be expressed as:

$$2 - k \leq W(E^-) \leq 2k - 1.$$

Based on the bounds of $W(E^+)$ and $W(E^-)$, the range of total weights $W(E)$ is given by:

$$2 - k \leq W(E) \leq 3k.$$

Next, we identify the smallest and largest terms in the arithmetic sequence used for edge labeling. The smallest term, U_1, is given as:

$$U_1 = 2 - k \implies a = 2 - k.$$

The largest term, U_n, corresponds to the upper limit:

$$U_n = 3k.$$

We then use the general form of an arithmetic sequence to calculate the n-th term, which corresponds to the number of edges $|E|$. The formula for the n-th term is:

$$U_n = a + (n - 1)b,$$

where b is the common difference of the sequence. Substituting $a = 2 - k$ and $b = 1$, the value of $U_{|E|}$ is:

$$U_{|E|} = (2 - k) + (|E| - 1).$$

Simplifying this expression gives:

$$U_{|E|} = |E| - k + 1.$$

Since the largest term in the sequence cannot exceed the maximum bound $3k$, we establish the inequality:

$$3k \geq |E| - k + 1.$$

Rearranging the terms yields:

$$4k \geq |E| + 1.$$

Thus, the lower bound for k can be expressed as:

$$k \geq \frac{|E| + 1}{4}.$$

However, since k must be an integer, we take the ceiling function of the division result. Therefore:

$$k \geq \left\lceil \frac{|E| + 1}{4} \right\rceil.$$

In conclusion, we have proven that the minimum k for which a graph $\mathbb{G}$ satisfies the total edge irregularity strength (tes($\mathbb{G}$)) under the condition that edge weights form an arithmetic sequence is:

$$tes(\mathbb{G}) \geq \left\lceil \frac{|E|+1}{4} \right\rceil .$$

This result demonstrates that the lower bound of k is directly dependent on the number of edges $|E|$ in the graph, while also accounting for the structure of the arithmetic sequence used for edge weights. □

Theorem 2. *Let $\mathbb{S}_n$ be a star signed graph with $E^+(\mathbb{S}_n) = \{xx_i; i \equiv 1(mod\ 2)\}$ and $E^-(\mathbb{S}_n) = \{xx_i; i \equiv 0(mod\ 2)\}$, then*

$$tes(\mathbb{S}_n) = \begin{cases} 2, & n \in \{4,5,6\}, \\ \left\lceil \frac{n+1}{4} \right\rceil + 1, & n \in \{3,7\}, \\ \frac{n}{2} - 1, & n \geq 8,\ n \equiv 0 \pmod 2, \\ \frac{n-1}{2} - 1, & n \geq 8,\ n \equiv 1 \pmod 2 \end{cases}$$

Proof. The set of vertices of the star signed graph is $\{x\} \cup \{x_i; 1 \leq i \leq n\}$ and the set of edges is $E^+(\mathbb{S}_n) \cup E^-(\mathbb{S}_n)$, where $E^+(\mathbb{S}_n) = \{xx_i; i \equiv 1 \pmod 2\}$ and $E^-(\mathbb{S}_n) = \{xx_i; i \equiv 0 \pmod 2\}$. Based on the set of vertices and edges, we have $|V(\mathbb{S}_n)| = n+1$ and $|E(\mathbb{S}_n)| = n$. To prove this theorem, we will divide it into 4 cases.

Case 1. $n \in \{4,5,6\}$.

The first step in proving the theorem in Case 1 is to determine the lower bound. Based on Theorem 1, we have $tes(\mathbb{S}_n) \geq \left\lceil \frac{|E(\mathbb{S}_n)|+1}{4} \right\rceil = \left\lceil \frac{n+1}{4} \right\rceil = 2$. Next, we will prove the upper bound $tes(\mathbb{S}_n) \leq 2$. Let $k = 2$. We define the vertex labeling function and edge labeling function as follows:

$$f(x) = 1.$$

$$f(x_i) = \begin{cases} 1 & \text{if } i \in \{1,4,6\}, \\ 2 & \text{if } i \in \{2,3,5\}. \end{cases}$$

$$f(xx_i) = \begin{cases} 1 & \text{if } i \in \{1,2,3,4\}, \\ 2 & \text{if } i \in \{5,6\}. \end{cases}$$

The vertex labeling function and edge labeling function above show that $\max\{\{f(v) \mid v \in V(\mathbb{S}_n)\} \cup \{f(xx_i) \mid xx_i \in E(\mathbb{S}_n)\}\} = 2$. Based on these two functions, we have the edge weight function as follows:

$$w(xx_i) = \begin{cases} \frac{i+1}{2} + 2, & \text{for } i \equiv 1 \pmod 2, \\ -\frac{i}{2} + 3, & \text{for } i \equiv 0 \pmod 2. \end{cases}$$

The edge weights of E from the total labeling f form consecutive integers ranging from $3 - \frac{n}{2}$ to $\frac{n}{2} + 3$ for $n \in \{4,6\}$, and from 1 to 5 for $n = 5$, with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{S}_n)$ is 2 for $n \in \{4,5,6\}$.

Case 2. $n \in \{3,7\}$.

Subcase 2.1 $n = 3$.

Based on Theorem 1, we have $k \geq \left\lceil \frac{n+1}{4} \right\rceil = 1$. Assume that $k = 1$. If $k = 1$, then

$$w(xx_1) = f(x) + f(x_1) + f(xx_1) = 1 + 1 + 1 = 3,$$

$$w(xx_3) = f(x) + f(x_3) + f(xx_3) = 1 + 1 + 1 = 3.$$

There exist edges with the same weight, namely the edges xx_1 and xx_3. This leads to a contradiction with the requirements of total edge irregularity strength. Based on this condition, we have $k \geq \left\lceil \frac{n+1}{4} + 1 \right\rceil = 2$. Next, we will prove that $k \leq \left\lceil \frac{n+1}{4} + 1 \right\rceil = 2$ by defining the vertex and edge labeling functions as follows:

$$f(x) = f(x_1) = 1,$$

$$f(x_2) = f(x_3) = 2,$$

$$f(xx_i) = 1; 1 \leq i \leq 3.$$

The function of the vertex label and the edge label above shows that $\max\{\{f(v) \mid v \in V(\mathbb{S}_n)\} \cup \{f(xx_i) \mid xx_i \in E(\mathbb{S}_n)\}\} = 2$. Based on these two functions, we have the edge weight function as follows:

$$w(xx_1) = 3, w(xx_2) = 2, w(xx_3) = 4.$$

The edge weights of E from the total labeling f form consecutive integers ranging from 2 to 4 for $n = 3$ with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{S}_n)$ is $\lceil \frac{n+1}{4} + 1 \rceil$ for $n = 3$.
Subcase 2.2 $n = 7$.
Based on Theorem 1, we have $k \geq \lceil \frac{n+1}{4} \rceil = 2$. If $f(x) = 1$, then $W(E^+) = \{3, 4, 5, 6\}$. Assume $k = 2$, then $1 + 2k = 6 \to 2k = 5 \to k = \frac{5}{2}$. This leads to a contradiction since k must be an integer. Therefore, we have $k \geq \lceil \frac{n+1}{4} + 1 \rceil = 3$. Next, we prove $k \leq \lceil \frac{n+1}{4} + 1 \rceil = 3$ by defining the vertex and edge labeling function as follows.

$$f(x) = 1.$$

$$f(x_i) = \begin{cases} 1, & \text{for } i \in \{1, 4, 6\}, \\ 2, & \text{for } i \in \{2, 3, 5\}, \\ 3, & \text{for } i = 7. \end{cases}$$

$$f(xx_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq 4, \\ 2, & \text{for } 5 \leq i \leq 6. \end{cases}$$

The vertex labeling function and the edge labeling function above show that $\max\{\{f(v) \mid v \in V(\mathbb{S}_n)\} \cup \{f(xx_i) \mid xx_i \in E(\mathbb{S}_n)\}\} = 3$. Based on these two functions, we have the edge weight function as follows:

$$w(xx_i) = \begin{cases} \frac{i+1}{2} + 2, & \text{for } i \equiv 1 \pmod 2, \\ -\frac{i}{2} + 3, & \text{for } i \equiv 0 \pmod 2. \end{cases}$$

The edge weights of E from the total labeling f form consecutive integers ranging from 0 to 6 for $n = 7$ with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{S}_n)$ is $\lceil \frac{n+1}{4} + 1 \rceil$ for $n = 7$.
Case 3. $n \geq 8$, $n \equiv 0 \pmod 2$.
The first step to proving this theorem is to determine the value of k. The minimum edge weight when $n \geq 8$, $n \equiv 0 \pmod 2$ is $3 - \frac{n}{2}$, while the formula for obtaining the smallest edge weight is $2 - k$. Based on this, we have $2 - k = 3 - \frac{n}{2} \to k = \frac{n}{2} - 1$. Let $k = \frac{n}{2} - 1$. We define the vertex labeling function and the edge labeling function as follows:

$$f(x) = 1.$$

$$f(x_i) = \begin{cases} \frac{i-1}{4} + 1 & \text{for } i \equiv 1 \pmod 4, \\ \frac{i+1}{4} + 1 & \text{for } i \equiv 3 \pmod 4, \\ 2 & \text{for } i = 2, \\ 1 & \text{for } i \geq 4,\ i \equiv 0 \pmod 2. \end{cases}$$

$$f(xx_i) = \begin{cases} \frac{i-1}{4} + 1 & \text{for } i \equiv 1 \pmod 4, \\ \frac{i+1}{4} & \text{for } i \equiv 3 \pmod 4, \\ 1 & \text{for } i = 2, \\ \frac{i}{2} - 1 & \text{for } i \geq 4,\ i \equiv 0 \pmod 2. \end{cases}$$

The vertex labeling function and the edge labeling function above show that $\max\{\{f(v) \mid v \in V(\mathbb{S}_n)\} \cup \{f(xx_i) \mid xx_i \in E(\mathbb{S}_n)\}\} = \frac{n}{2} - 1$. Based on these two functions, we have the edge weight function as follows:

$$w(xx_i) = \begin{cases} \frac{i+1}{2} + 2, & \text{for } i \equiv 1 \pmod 2, \\ -\frac{i}{2} + 3, & \text{for } i \equiv 0 \pmod 2. \end{cases}$$

The edge weights of E from the total labeling f form consecutive integers ranging from $3 - \frac{n}{2}$ to $\frac{n}{2} + 2$ for $n \geq 8$, $n \equiv 0 \pmod 2$ with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{S}_n)$ is $\frac{n}{2} - 1$ for $n \geq 8$, $n \equiv 0 \pmod 2$.
Case 4. $n \geq 8$, $n \equiv 1 \pmod 2$
The first step to prove this theorem is to determine the value of k. The minimum edge weight when $n \geq 8$, $n \equiv 1 \pmod 2$ is $3 - \frac{n-1}{2}$, while the formula for obtaining the smallest edge weight is $2 - k$ since $f(x) = 1$ and $f(xx_i) = 1$. Based on this, we have $2 - k = 3 - \frac{(n-1)}{2} \implies k = \frac{(n-1)}{2} - 1$. Let $k = \frac{(n-1)}{2} - 1$. We define the vertex labeling function and the edge labeling function as follows:

$$f(x) = 1.$$

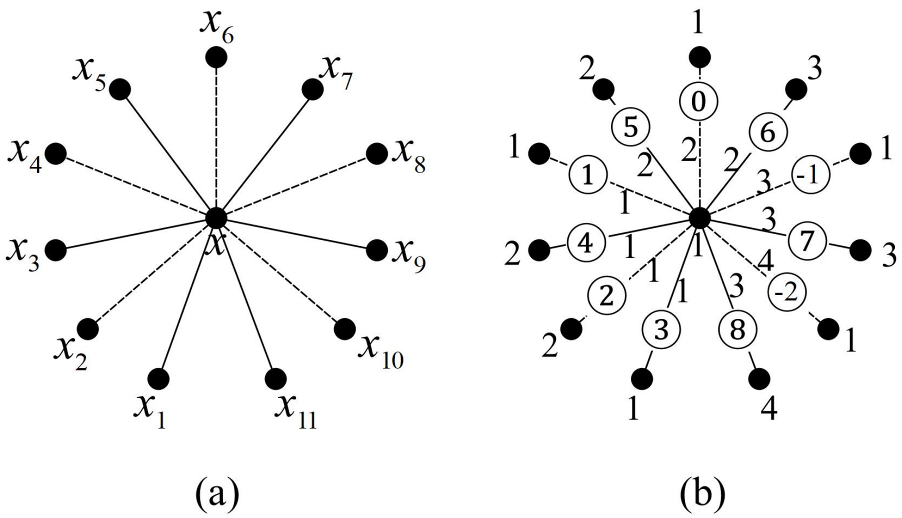

FIGURE 1. (a) Star signed graph ($\mathbb{S}_{11}$), (b) $tes(\mathbb{S}_{11}) = 4$.

$$f(x_i) = \begin{cases} \frac{i-1}{4} + 1 & \text{for } i \equiv 1 \pmod 4, \\ \frac{i+1}{4} + 1 & \text{for } i \equiv 3 \pmod 4, \\ 2 & \text{for } i = 2, \\ 1 & \text{for } i \geq 4,\ i \equiv 0 \pmod 2. \end{cases}$$

$$f(xx_i) = \begin{cases} \frac{i-1}{4} + 1 & \text{for } i \equiv 1 \pmod 4, \\ \frac{i+1}{4} & \text{for } i \equiv 3 \pmod 4, \\ 1 & \text{for } i = 2, \\ \frac{i}{2} - 1 & \text{for } i \geq 4,\ i \equiv 0 \pmod 2. \end{cases}$$

The vertex labeling function and the edge labeling function above show that $\max\{\{f(v) \mid v \in V(\mathbb{S}_n)\} \cup \{f(xx_i) \mid xx_i \in E(\mathbb{S}_n)\}\} = \frac{(n-1)}{2} - 1$. Based on these two functions, we have the edge weight function as follows:

$$w(xx_i) = \begin{cases} \frac{i+1}{2} + 2, & \text{for } i \equiv 1 \pmod 2, \\ -\frac{i}{2} + 3, & \text{for } i \equiv 0 \pmod 2. \end{cases}$$

The edge weights of E from the total labeling f form consecutive integers ranging from $3 - \frac{n-1}{2}$ to $\frac{n+1}{2} + 2$ for $n \geq 8$, $n \equiv 1 \pmod 2$ with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{S}_n)$ is $\frac{n-1}{2} - 1$ for $n \geq 8$, $n \equiv 1 \pmod 2$. The illustration of the edge irregular total $k-$labeling of $\mathbb{S}_n$ can be seen in Figure 1.

□

Theorem 3. *Let $\mathbb{L}_n$ be a ladder signed graph with $E^+(\mathbb{L}_n) = \{a_i a_{i+1}, b_i b_{i+1}; 1 \leq i \leq n-1\} \cup \{a_i b_i; i \equiv 1(mod\ 2)\}$, $E^-(\mathbb{L}_n) = \{a_i b_i; i \equiv 0(mod\ 2)\}$ and $n \equiv 1(mod\ 2)$, then*

$$tes(\mathbb{L}_n) = \left\lceil \frac{5n+1}{6} \right\rceil$$

Proof. The set of vertices of the ladder signed graph is $\{a_i b_i; 1 \leq i \leq n\}$ and the set of edges is $E^+(\mathbb{L}_n) \cup E^-(\mathbb{L}_n)$, where $E^+(\mathbb{L}_n) = \{a_i a_{i+1}, b_i b_{i+1}; 1 \leq i \leq n-1\} \cup \{a_i b_i; i \equiv 1(\text{mod } 2)\}$ and $E^-(\mathbb{L}_n) = \{a_i b_i; i \equiv 0(\text{mod } 2)\}$. Based on the set of vertices and edges, we have $|V(\mathbb{L}_n)| = 2n$ and $|E(\mathbb{L}_n)| = 3n-2$. To prove this theorem, we will establish both the lower and upper bounds. First, we will discuss the lower bound of k. In the topic of edge irregular total k-labeling on $\mathbb{L}_n$, the resulting edge weights must be distinct and form an arithmetic sequence. Since $|E^+(\mathbb{L}_n)| \geq E^-(\mathbb{L}_n)$, we use the possible weights from $E^+(\mathbb{L}_n)$ to obtain the lower bound. We have $|E^+(\mathbb{L}_n)| = \frac{5n-3}{2}$, the smallest edge weight is 3, and the largest edge weight is

$$U_{|E^+(\mathbb{L}_n)|} = a + (|E^+(\mathbb{L}_n)| - 1)b = 3 + \left(\frac{5n-3}{2} - 1\right) \cdot 1 = \frac{5n+1}{2}.$$

The largest edge weight from $E^+(\mathbb{L}_n)$ can also be obtained from $3k$, hence $3k \geq \frac{5n+1}{2} \rightarrow k \geq \lceil \frac{5n+1}{6} \rceil$.

Next, we will prove the upper bound of k. To prove this theorem, we will divide it into several cases. These cases will also describe the vertex labeling function, edge labeling function, and the edge weight function for each respective case.

Case 1. $n = 3$

The lower bound of k is given by $k \geq \lceil \frac{5n+1}{6} \rceil$. Now, we substitute $n = 3$ into the lower bound so that we have $k \geq \lceil \frac{5\times3+1}{6} \rceil = 3$. Next, we will prove the upper bound of k, which is $k \leq 3$, by defining the vertex labeling function and the edge labeling function as follows:

$$f(a_i) = 1 \text{ for } 1 \leq i \leq 3,$$
$$f(b_i) = 3 \text{ for } 1 \leq i \leq 3$$
$$f(a_i a_{i+1}) = f(b_i b_{i+1}) = i \text{ for } 1 \leq i \leq 2,$$
$$f(a_i b_i) = \begin{cases} 1 & \text{for } i = 1 \\ 2 & \text{for } 2 \leq i \leq 3 \end{cases}$$

The vertex labeling function and the edge labeling function above show that $\max\{\{f(a_i), f(b_i) \mid a_i, b_i \in V(\mathbb{L}_n)\} \cup \{f(a_i a_{i+1}), f(a_i b_i), f(b_i b_{i+1}) \mid a_i a_{i+1}, a_i b_i, b_i b_{i+1} \in E(\mathbb{L}_n)\}\} = 3$. Based on these two functions, we have the edge weight function as follows:

$$w(a_i a_{i+1}) = i + 2 \text{ for } i \in \{1, 2\},$$
$$w(a_1 b_1) = 5, w(a_2 b_2) = 2, w(a_3 b_3) = 6,$$
$$w(b_i b_{i+1}) = i + 6 \text{ for } i \in \{1, 2\}.$$

The edge weights of E from the total labeling f form consecutive integers ranging from 2 to 8 for $n = 3$, with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{L}_n)$ is 3 for $n = 3$.

Case 2. $n = 5$

The lower bound of k is given by $k \geq \lceil \frac{5n+1}{6} \rceil$. Now, we substitute $n = 5$ into the lower bound so that we have $k \geq \lceil \frac{5\times5+1}{6} \rceil = 5$. Next, we will prove the upper bound of k, which is $k \leq 5$, by defining the vertex labeling function and the edge labeling function as follows:

$$f(a_i) = 1 \text{ for } 1 \leq i \leq 5$$
$$f(b_i) = 4 \text{ for } 1 \leq i \leq 5$$
$$f(a_i a_{i+1}) = i \text{ for } 1 \leq i \leq 4$$
$$f(a_i b_i) = \begin{cases} 2 & \text{for } i = 1 \\ 3 & \text{for } i \in \{2, 3\} \\ 4 & \text{for } i \in \{4, 5\} \end{cases}$$
$$f(b_i b_{i+1}) = i + 1 \text{ for } 1 \leq i \leq 4$$

The vertex labeling function and the edge labeling function above show that $\max\{\{f(a_i), f(b_i) \mid a_i, b_i \in V(\mathbb{L}_n)\} \cup \{f(a_i a_{i+1}), f(a_i b_i), f(b_i b_{i+1}) \mid a_i a_{i+1}, a_i b_i, b_i b_{i+1} \in E(\mathbb{L}_n)\}\} = 5$. Based on these two functions, we have the edge weight function as follows:

$$w(a_i a_{i+1}) = i + 2 \text{ for } 1 \leq i \leq 4,$$
$$w(a_1 b_1) = 7, w(a_2 b_2) = 2, w(a_3 b_3) = 8, w(a_4 b_4) = 1, w(a_5 b_5) = 9,$$
$$w(b_i b_{i+1}) = i + 9 \text{ for } 1 \leq i \leq 4.$$

The edge weights of E from the total labeling f form consecutive integers ranging from 1 to 13 for $n = 5$, with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{L}_n)$ is 5 for $n = 5$.

Case 3. $n = 7$

The lower bound of k is given by $k \geq \lceil \frac{5n+1}{6} \rceil$. Now, we substitute $n = 7$ into the lower bound so that we have $k \geq \lceil \frac{5\times7+1}{6} \rceil = 6$. Next, we will prove the upper bound of k, which is $k \leq 6$, by defining the vertex labeling function and the edge labeling function as follows:

$$f(a_i) = \begin{cases} 1, & \text{for } i = 1 \text{ or } i \equiv 0 (\text{mod } 2), 2 \leq i \leq 7 \\ 2, & \text{for } i \equiv 1 (\text{mod } 2), 3 \leq i \leq 7 \end{cases} \qquad f(b_i) = \begin{cases} 5, & \text{for } i \in \{1, 2\} \\ 6, & \text{for } 3 \leq i \leq 7 \end{cases}$$
$$f(a_i a_{i+1}) = \begin{cases} 1, & \text{for } i = 1 \\ i - 1, & \text{for } 2 \leq i \leq 6 \end{cases} \qquad f(b_i b_{i+1}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \\ i, & \text{for } 4 \leq i \leq 6 \end{cases}$$

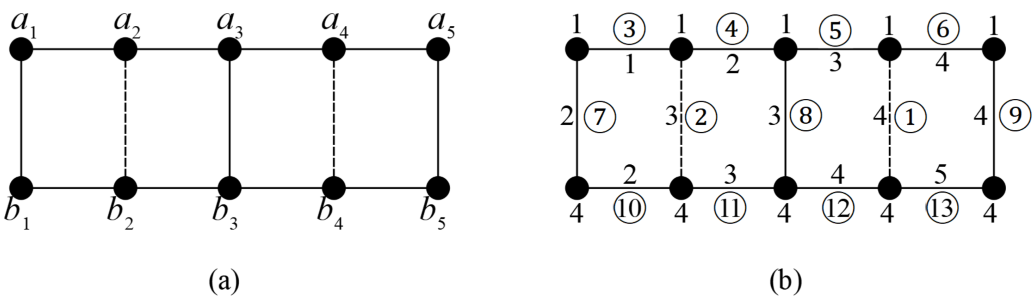

FIGURE 2. (a) Ladder signed graph ($\mathbb{L}_5$), (b) $tes(\mathbb{S}_5) = 5$.

$$f(a_ib_i) = \begin{cases} 2, & \text{for } i = 3 \\ 3, & \text{for } i \in \{1, 5\} \\ 4, & \text{for } i = 7 \\ 5, & \text{for } i = 6 \\ 6, & \text{for } i \in \{2, 4\} \end{cases}$$

The vertex labeling function and the edge labeling function above show that $\max\{\{f(a_i), f(b_i) \mid a_i, b_i \in V(\mathbb{L}_n)\} \cup \{f(a_ia_{i+1}), f(a_ib_i), f(b_ib_{i+1}) \mid a_ia_{i+1}, a_ib_i, b_ib_{i+1} \in E(\mathbb{L}_n)\}\} = 6$. Based on these two functions, we have the edge weight function as follows:

$$w(a_ia_{i+1}) = i + 2 \text{ for } 1 \le i \le 6,$$
$$w(a_1b_1) = 9, w(a_2b_2) = 0, w(a_3b_3) = 10, w(a_4b_4) = 1, w(a_5b_5) = 11, w(a_6b_6) = 2, w(a_7b_7) = 12,$$
$$w(b_ib_{i+1}) = i + 12 \text{ for } 1 \le i \le 6.$$

The edge weights of E from the total labeling f form consecutive integers ranging from 0 to 18 for $n = 7$, with no two edges having the same weight. Thus, the total edge strength $tes(\mathbb{L}_n)$ is 6 for $n = 7$.

Case 4. $n \equiv 1 (\text{mod } 2), 9 \le i \le n$.

The lower bound of k is given by $k \ge \left\lceil \frac{5n+1}{6} \right\rceil$. Now, we will prove the upper bound of k, which is $k \le \left\lceil \frac{5n+1}{6} \right\rceil$, by defining the vertex labeling function and the edge labeling function as follows:

$$f(a_i) = \begin{cases} 1, & \text{for } i = 1, \\ & \text{for } i \equiv 0 \pmod 2, 2 \le i \le n-1 \\ 2, & \text{for } i \equiv 1 \pmod 2, 3 \le i \le n-2 \\ n-k, & \text{for } i = n \end{cases}$$

$$f(b_i) = \begin{cases} k & \text{for } n-4 \le i \le n, \\ k + \frac{i-n}{2} + 2 & \text{for } i \equiv 1 \pmod 2, 1 \le i \le n-5 \\ k + \frac{i-n+3}{2} & \text{for } i \equiv 0 \pmod 2, 2 \le i \le n-6 \end{cases}$$

$$f(a_ia_{i+1}) = \begin{cases} 1, & \text{for } i = 1, \\ i-1, & \text{for } 2 \le i \le n-2 \\ k. & \text{for } i = n-1 \end{cases}$$

$$f(a_ib_i) = \begin{cases} k, & \text{for } i \equiv 0 \pmod 2, 2 \le i \le n-3, \\ k-6, & \text{for } i \equiv 1 \pmod 2, 3 \le i \le n-5, \\ k-5, & \text{for } i = n-2 \text{ or } i = 1, \\ k-1, & \text{for } i = n-1, \\ \frac{n+3}{2}, & \text{for } i = n. \end{cases}$$

$$f(b_ib_{i+1}) = \begin{cases} \frac{5n+1}{2} - 2k & \text{for } i = n-1, \\ \frac{5n-1}{2} - 2k & \text{for } i = n-2, \\ \frac{5n-3}{2} - 2k & \text{for } i = n-3, \\ \frac{5n-5}{2} - 2k & \text{for } 1 \le i \le n-4. \end{cases}$$

In the vertex labeling and edge labeling functions above, the maximum label is k even though there exists a labeling function with the value $\frac{n+3}{2}$, because $\frac{n+3}{2} \leq \lceil \frac{5n+1}{6} \rceil = k$. The vertex labeling function and the edge labeling function above show that $\max\{\{f(a_i), f(b_i) \mid a_i, b_i \in V(\mathbb{L}_n)\} \cup \{f(a_ia_{i+1}), f(a_ib_i), f(b_ib_{i+1}) \mid a_ia_{i+1}, a_ib_i, b_ib_{i+1} \in E(\mathbb{L}_n)\}\} = k$. Based on these two functions, we have the edge weight function as follows:

$$w(a_ia_{i+1}) = i + 2 \text{ for } 1 \leq i \leq n-1,$$
$$w(a_ib_i) = \begin{cases} n + \frac{i+3}{2}, & \text{for } i \equiv 1 \pmod 2, 1 \leq i \leq n, \\ \frac{i+5-n}{2}, & \text{for } i \equiv 0 \pmod 2, 2 \leq i \leq n-1, \end{cases}$$
$$w(b_ib_{i+1}) = \frac{3n+3}{2} + i \text{ for } \leq i \leq n-1.$$

The edge weights of E from the total labeling f form consecutive integers ranging from $k-5$ to $\frac{5n+1}{2n}$ for $n \equiv 1 (\text{mod } 2), 9 \leq i \leq n$, with no two edges have the same weight. Thus, the total edge strength $tes(\mathbb{L}_n)$ is $\lceil \frac{5n+1}{6} \rceil$ for $n \equiv 1 (\text{mod } 2), 9 \leq i \leq n$.

Based on Case 1 through Case 4, it has been proven that $k \leq \lceil \frac{5n+1}{6} \rceil$. Therefore, we have both the lower and upper bounds of $tes(\mathbb{L}_n)$, namely $\lceil \frac{5n+1}{6} \rceil \leq tes(\mathbb{L}_n) \leq \lceil \frac{5n+1}{6} \rceil$. It is thus proven that $tes(\mathbb{L}_n) = \lceil \frac{5n+1}{6} \rceil$ for $n \equiv 1 \pmod 2$, with $n \geq 3$. □

3.2. **Conceptual Framework Design for Watermarking.** This subsection outlines the developed CFD that combines edge irregular total k-labeling of signed graphs with asymmetric cryptography, forming the foundation for a secure and verifiable image watermarking system. The Conceptual Framework Design (CFD) developed in this study integrates asymmetric cryptography with edge irregular total k-labeling of signed graphs to construct a robust and verifiable image watermarking system. The workflow is divided into a series of sequential steps that systematically bridge mathematical graph labeling with practical digital watermark embedding. The CFD can be seen in Figure 3.

The process begins with the determination of the signed graph structure (Step 1), where vertices and signed edges are defined according to the desired graph topology. In Step 2, vertex and edge labeling is manually assigned based on predefined labeling functions. The edge weights are then calculated in Step 3 by summing the vertex and edge labels. Step 4 ensures that all edge weights are unique and satisfy the condition for edge irregular total k-labeling; if the condition is not met, relabeling is conducted manually.

After validation, Step 5 formulates the labeling and weight functions used throughout the system. Image processing begins in Step 6 with input and preprocessing, where the image is converted to grayscale and divided into sub-blocks (typically 8×8 pixels). In Step 7, the number of sub-blocks is counted and compared with the graph size to ensure that the adjacency matrix size is sufficient.

Next, Step 8 involves converting the edge weights of the signed graph into an adjacency matrix, which is then transformed into binary format in Step 9. At this point, two binary matrices are available: one from the image and one from the graph. In Step 10, these matrices are embedded using techniques such as LSB or transform-domain embedding, and the result is flattened into a one-dimensional bitstream W.

In Step 11, a cryptographic hash function (e.g., SHA-256) is applied to W, resulting in a digital digest H. This digest is then encrypted using a private key to generate a digital signature σ in Step 12. The signature is embedded back into a specific part of the image (Step 13), serving as a cryptographic proof of integrity. Finally, Step 14 outputs the watermarked image, which contains a digital signature that can later be verified using the public key to confirm the authenticity and integrity of the content.

This CFD not only establishes a secure end-to-end watermarking scheme, but also demonstrates a strong connection between theoretical graph labeling and practical information security applications.

3.2.1. *System Model and Notations.* Let $I \in \{0, \ldots, 255\}^{M \times N}$ be the grayscale cover image. The image is partitioned into non-overlapping blocks

$$B_i \in \{0, \ldots, 255\}^{b \times b}, \qquad i = 1, \ldots, N_b,$$

where b is the block size and $N_b = (M/b)(N/b)$.

Let $\Sigma = (G, \sigma)$ be a signed graph, where $G = (V, E)$ and $\sigma : E \to \{-1, +1\}$ assigns a sign to each edge. A TES labeling is a function $f : V \cup E \to \{1, \ldots, k\}$ which induces unique edge weights

$$w(uv) = f(u) + f(v) + \sigma(uv)\, f(uv), \qquad uv \in E, \tag{1}$$

and ensures $w(e_i) \neq w(e_j)$ whenever $e_i \neq e_j$.

Let $W = (w(e_1), \ldots, w(e_{|E|}))$ be the lexicographically ordered vector of edge weights. Each $w(e_i)$ is encoded into an 8-bit binary string and concatenated to form the bitstring

$$K_g = \text{bin}(w(e_1)) \,\|\, \text{bin}(w(e_2)) \,\|\, \cdots \,\|\, \text{bin}(w(e_{|E|})).$$

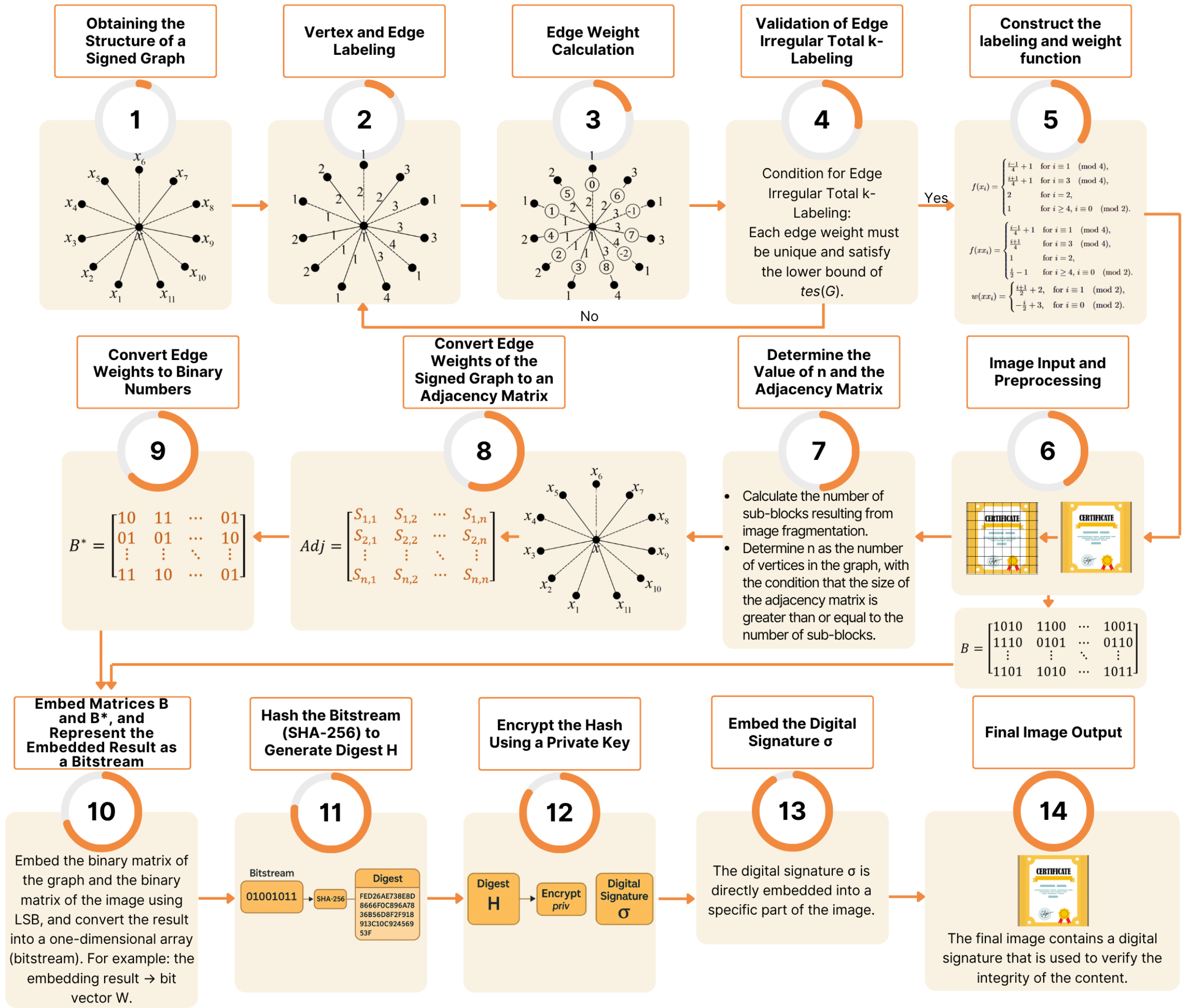

FIGURE 3. CFD of the image watermarking system based on asymmetric cryptography and signed graphs.

From K_g, two auxiliary keys are derived:

$$K_s = K_g[1 : l_s], \qquad K_{\text{seed}} = K_g[l_s + 1 : l_s + l_p],$$

where K_s is the shift-key controlling block-index permutations, and K_{seed} is the seed for a PRNG that determines pixel-level embedding positions.

The asymmetric key pair (pk, sk) is generated independently using a standard scheme (e.g., RSA or ECC). The graph does not produce the public/private key; instead, it provides auxiliary randomness for the embedding domain.

3.2.2. *Graph-Based Auxiliary Key Generation.* The generation of K_s and K_{seed} from TES labeling is described in Algorithm 1. The uniqueness of $w(uv)$ guaranteed by TES implies that K_g and its derived keys are deterministic but uniformly distributed over the set of feasible labelings.

Algorithm 1 Graph-Based Auxiliary Key Derivation

Require: Signed graph $\Sigma = (G, \sigma)$ with TES labeling f
Ensure: Shift-key K_s, seed K_{seed}
1: Compute all edge weights using (1)
2: Sort edges lexicographically to obtain $W = (w(e_1), \dots, w(e_{|E|}))$
3: Encode each $w(e_i)$ into 8-bit binary and concatenate to form K_g
4: Extract K_s and K_{seed} from consecutive segments of K_g
5: **return** K_s, K_{seed}

3.2.3. *Watermark Embedding Algorithm.* The watermark payload includes (i) the adjacency matrix A or an encoded variant, and (ii) the digital signature $\sigma = \mathsf{Enc}_{sk}(H)$, where $H = \text{SHA256}(A)$.

The block-level and pixel-level embedding positions are not chosen sequentially, but are pseudo-randomly determined by two graph-derived keys. First, the shift key K_s is mapped to a block permutation π_{K_s} that reorders the set of image blocks and thus specifies the visiting order at the block level. Then, within each selected block, a pseudo-random number generator (PRNG) is initialized with the seed K_{seed} to produce an index sequence indicating which pixels (and, if needed, which bit-planes) are used to carry the watermark bits. In this way, the same cover image can give rise to different embedding patterns depending on the underlying labeled graph, and an adversary who does not know K_s and K_{seed} cannot predict or reliably reproduce the block and pixel positions used for embedding. Algorithm 2 describes the detailed embedding process.

Algorithm 2 Watermark Embedding

Require: Cover image I, keys K_s, K_{seed}, private key sk
Ensure: Watermarked image I_w

```
Partition I into blocks {B_i}_{i=1}^{N_b}
Construct adjacency matrix A from (G, σ, f)
Compute H = SHA256(A) and σ = Enc_sk(H)
Form the watermark bitstream M = bin(A)||bin(σ)
Generate block permutation π_{K_s}
Initialize PRNG with seed K_seed
for each bit m_j of M do
    Select block B_{π_{K_s}(i)}
    Select pixel position (x, y) using PRNG
    Modify LSB of B_{π_{K_s}(i)}(x, y) to embed m_j
end for
Reassemble blocks into I_w
return I_w
```

3.2.4. *Watermark Extraction and Signature Verification.* Algorithm 3 describes the extraction process. The public key pk is used to verify the authenticity of the watermark payload.

Algorithm 3 Watermark Extraction and Verification

Require: Watermarked image I_w, keys K_s, K_{seed}, public key pk
Ensure: Authenticity decision

```
Partition I_w into blocks and regenerate π_{K_s}
Initialize PRNG with K_seed
Extract bitstream M'
Parse M' into adjacency matrix A' and signature σ'
Compute H' = SHA256(A')
Verify whether Dec_pk(σ') = H'
if valid then
    Accept: the image is authentic
else
    Reject: the image is tampered
end if
```

3.2.5. *Security Analysis.*
Unforgeability. Even if an attacker understands TES labeling, they cannot generate a valid signature σ because it requires the private key sk. Any modification to the embedded adjacency matrix A produces a different hash H', causing signature verification to fail.
Key-space and randomness. TES guarantees that every edge weight is distinct, ensuring K_g contains no duplicate patterns. The permutation entropy of K_s and the PRNG space from K_{seed} collectively provide sufficient randomness for secure embedding positions.
Tamper sensitivity (Fragile Watermarking). Because LSB embedding is used, modifications such as filtering, cropping, or compression will alter the extracted A' and invalidate the signature, making the system suitable for authentication rather than robust watermarking.

3.2.6. *Complexity Analysis.* Computing TES weights requires $O(|E|)$ time. Constructing the adjacency matrix takes $O(|V|^2)$ or $O(|E|)$ for sparse graphs. Embedding LSB bits requires $O(MN)$ operations. The entire framework therefore operates in linear time with respect to image size, with negligible overhead compared to standard LSB watermarking.

4. **Concluding Remarks.** In this paper, we have explored the edge irregular total k-labeling of signed graphs, with a particular focus on determining the total edge irregularity strength $\text{tes}(\mathbb{G})$ for specific classes of signed graphs. We successfully established the sharpest lower bounds for $\text{tes}(\mathbb{G})$ and computed exact values for several families, including star and ladder signed graphs. These results enrich the theoretical landscape of graph labeling and contribute significantly to the combinatorial structure of signed graphs.

In addition to these theoretical findings, this study introduces a Conceptual Framework Design (CFD) that integrates edge irregular total k-labeling with asymmetric cryptography to construct a secure and verifiable image watermarking system. The CFD operationalizes the labeling results by generating public, private, and shift keys, and embedding the labeling structure into binary representations of images. The framework incorporates cryptographic hashing and digital signature techniques, demonstrating the practical applicability of signed graph theory in the field of information security.

Despite these advances, determining the total edge irregularity strength for arbitrary signed graphs remains a complex and unresolved problem, particularly for graphs of large or undefined order. This challenge is believed to be computationally difficult, and possibly NP-hard. To encourage further research, we propose the following open problems:

(i) Determine the exact values of $\text{tes}(\mathbb{G})$ under various graph operations on signed graphs.
(ii) Characterize the conditions for the existence of edge irregular total k-labelings in signed graphs with specific structural constraints.
(iii) Establish the exact value of $\text{tes}(\mathbb{L}_n)$, where $\mathbb{L}_n$ denotes the ladder signed graph of order n, especially for even values of n.

These open problems highlight promising directions for future investigations into the structural, algorithmic, and applied aspects of signed graph labelings.

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