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A New Compressed Sensing Algorithm Design Based on Wavelet Frame and Dictionary

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Abstract. *Compressed sensing has been paid a lot of attention for its contribution for image restoration, image reconstruction and image representation. Two most common research orientations are the basic theory research and the application research respectively. A novel design for compressed sensing frame based on the wavelet frame and dictionary is proposed in this paper. It belongs to the basic theory research and the good performance in the experiments show its efficiency.*
Keywords: Compressed Sensing, Wavelet frame, Sparse Representation, Dictionary

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1. **Introduction.** The compressed sensing methods are proposed for improving the efficiency of the classical Nyquist sampling method. Its main work is to exploit sparsity in signal processing. As we all know, a spare signal is one that has few non-zero entries relative to its dimension. A lot of signals are either sparse in their original form or can be represented as a sparse signal in a transform domain. And the spare property is very important for image compression and restoration, echo cancellation, channel equalization, and so on. In traditional view, signals are first sampled at Nyquist rate or more and then compressed for efficient storage and/or transmission. In Compressed Sensing (CS) view, the signal is acquired directly in compressed form. Some early work has been done on this topic such as the reference [1-7]. The main work for CS is to design a proper matrix named sensing matrix which is combined with the transform matrix and the projection matrix [8-10]. So, various transforms are introduced in this research. Wavelet analysis of good time-frequency characteristics widely used in the image compression field has become one of the mainstream technologies, which has the high decorrelation and energy compression efficiency, and can effectively remove the blocking effect and mosquito noise [11,12,13]. It has been used in the CS research for all its good properties [14,15,16]. In this paper, we present a novel wavelet based CS fame. Section 2 will give some related knowledges about the compressed sensing and wavelet frame. The proposed design will be introduced in the section 3. Then the experimental results will be shown in the section 4. The conclusion will be given in the section 5. At last, the reference will be listed.

2. **Related Work.**

2.1. **Compressed Sensing.** The basic process for CS can be shown as below. For a finite dimensional signal$x \in R^{n*1}$ CCA is multivariate statistical analysis which study two groups of random variable relationship between the statistical method [10]. Based on the idea of CCA, we set up the correlation criterion function between two groups of feature vectors, and calculate typical projection vector set of the two groups according to the criterion. Then combined canonical correlation characteristics are extracted for recognition. The framework of the multimodal biometric algorithm as figure 1.

2.2. **Compressed Sensing.** The basic process for CS can be shown as below.

For a finite dimensional signal $\in R^{N\times 1}$, suppose any signal in R^N space can be expressed by a normal orthogonal base of this space $\psi = [\varphi_1, \varphi_2, \cdots, \varphi_N]$, then we have the following linear combinations expression.

$$x = \sum_{i=1}^{N} \varphi_i \alpha_i = \psi\alpha \tag{1}$$

The above formula can be called either as the equivalent representations of the signal x or the linear decomposition. $\alpha =< x, \varphi_i >$ is the projection coefficients and $\alpha = \psi^T x$ is a projection vector. $\psi = [\varphi_1, \varphi_2, \cdots, \varphi_N]$ is an orthogonal matrix, α and x and are two vectors with $N \times 1-$dimension. ψ is a $N \times N$ matrix. The CS theory believes that if only the non-zero coefficients number of the transformed signal $K << N$ then x is sparse on the base ψ or it can be compressed. The signal x is called $K-$ sparse. If the formula (1) is some kind of sparse representation for the signal x, then ψ can be called a sparse base.

If a signal satisfies the sparse condition, it can be projected into an observation matrix space and the projection values are called the observation sequence recorded as y. This process can be shown as below.

$$y = \Phi x \tag{2}$$

Here, Φ is a $N \times N$ matrix and y is a $N \times 1$ column vector. As we can see from the above formula (2), the sequence y after the observation process has a smaller dimension value M than the original signals N which means that the signal has already been compressed and the data after compression is much smaller than the Nyquist sampling data. So a compression ratio definition can be given as below.

$$q = M/N \tag{3}$$

If we bring the formula (1) to the (2), then we can get the following expression (4).

$$y = \Phi x = \Phi\psi\alpha = \Delta\alpha \tag{4}$$

Then matrix $\Delta = \Phi\psi$ can be called the sensing matrix. The compressed sensing theory believes that if the observation matrix satisfies the Restricted Isometry Property condition and the Irrelevant Characteristics [17], and the signal x is also a $K-$ Sparse $(K < M << N)$, then we can get the sparse coefficient x through solving a zero normal optimization problem shown below.

$$\min ||\alpha||_0 \; s.t. \; y = \Delta\alpha \tag{5}$$

Then you can get the signal x using the formula (1).

As we can see from above, the traditional sampling compression and compressed sensing process can be shown in the following figure 1 (a) and (b).

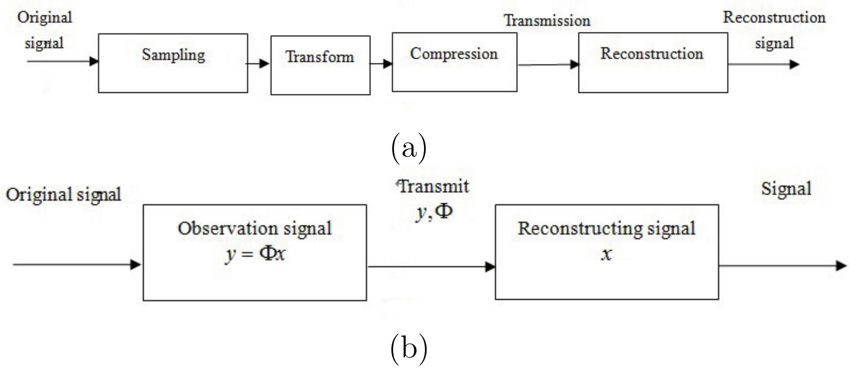

FIGURE 1. Traditional Sampling Compression Process (a) and the Compressed Sensing Process (b)

2.3. **Wavelet.** As an efficient tool for signal processing, wavelet has been widely used in many research areas.

If we let

$$\phi_{j,k}(x) := 2^{\frac{j}{2}}\psi(2^j x - k), \ j,k \in Z \tag{6}$$

Then, the wavelet transform can be expressed as

$$(W_\psi f)(\frac{k}{2^j}, \frac{1}{2^j}) =< f, \psi_{j,k} > \tag{7}$$

Generally speaking, every function $f \in L^2(R)$ can be expressed as the following formula (8).

$$f(x) = \sum_{j,k=-\infty}^{\infty} < f, \psi_{j,k} > \psi^{j,k}(x) \tag{8}$$

Though the coefficients are values of the integral wavelet transform of relative to ψ, the series is not necessarily a wavelet series. To qualify as a wavelet series, there must exist some function $\tilde{\psi} \in L^2(R)$, such that the dual basis $\{\psi^{j,k}\}$ in the above series is obtained from $\tilde{\psi}$ by

$$\psi^{j,k}(x) = \tilde{\psi}_{j,k}(x) \tag{9}$$

here, as usual, the notation

$$\tilde{\psi}_{j,k}(x) := 2^{\frac{j}{2}}\tilde{\psi}(2^j x - k) \tag{10}$$

is used.

The reason for wavelet can be used in the CS is that it produces a large number of values having zero, or near zero, magnitudes. In fact, there have been some CS works based on the wavelet proposed recently [18,19].

2.4. **Wavelet Frame.** Different from the wavelet, the wavelet frame can be seen as a redundant base for the space $L^2(R)$ which can be defined as follows. Here the tight wavelet frame is considered.

The space $L^2(R)$ is the set of all the functions $f(x)$ which satisfies the following condition.

$$||f||_{L^2(R)} := (\int_R |f(x)|^2 dx)^{\frac{1}{2}} < \infty \tag{11}$$

And similar, $l_2(Z)$ is the set of all sequences defined on Z satisfying the following condition.

$$||h||_{l_2(Z)} := (\sum_{k \in Z} |h[k]|^2 dx)^{\frac{1}{2}} < \infty \tag{12}$$

For any function $f \in L^2(R)$, the dyadic dilation operator D is defined by the following formula (13).

$$Df(x) := \sqrt{2}f(2x) \tag{13}$$

And for $a \in R$, the translation operator can be defined as below.

$$T_a f(x) := f(x-a) \tag{14}$$

Then, we have$T_a D^j = D^j T_{2^j a}$. For some given $\Psi := \{\psi_1, \cdots, \psi_r\} \subset L^2(R)$, a wavelet system can be defined as

$$X(\Psi) := \{\psi_{i,j,k} : 1 \leq l \leq r; j,k \in Z\} \tag{15}$$

here $\phi_{i,j,k} = D^j T_k \psi_l = 2^{\frac{j}{2}} \psi_l(2^j - k)$. The system $X(\Psi) \leq L^2(R)$ is called a tight wavelet frame of $L^2(R)$ if

$$||f||^2_{L^2(R)} = \sum_{geX(\Psi)} | < f, g > |^2 \tag{16}$$

holds for all $f \in L^2(R)$, where $< \cdot, \cdot >$ is the inner product in $L^2(R)$.

Actually, when $X(\Psi)$ forms an orthonormal basis of $L^2(R)$, then $X(\Psi)$ is called an orthonormal wavelet basis. And when $X(\Psi)$ forms a tight frame of $L^2(R)$, then the $X(\Psi)$ is called a tight wavelet frame. Here we have a theorem as follows.

Theorem 2.1. *The wavelet system $Xf(\Psi)$ is a tight frame of $L^2(R)$ if and only if the identities*

$$\sum\sum |\hat{\psi}(2^k w)|^2 = 1; \sum_{\psi \in \Psi} \sum_{k=0}^{\infty} \hat{\psi}(2^k w) \overline{\hat{\psi}(2^k(w + (2j+1)2\pi))} = 0 \; j \in Z \tag{17}$$

And the theory of fames has a long history of the development even before the discovery of the multiresolution analysis of [20] and the systematic construction of compactly supported orthonormal wavelets of [21].

2.5. **Dictionary.** In fact, the definition for dictionary is something similar to the base for a space. Let Φ be a sequence of vectors $\Phi = \{\phi_i\}_{i=1}^N, N \geq M$ and the vectors need not to be linear independence, then any vector X in this space can be expressed as follows.

$$X = \Phi\alpha = [\phi_1 \phi_2 \cdots \phi_N] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \phi_i \tag{18}$$

Here, the coefficients $\alpha_i =< X, \phi_i >$ is the projection coefficients of the vector X onto the element ϕ_i. Normally, the combined matrix Φ is a $M \times M$ square matrix. When the vectors ϕ_i are not linear independent and $N >> M$, then the matrix Φ will be called a dictionary matrix and the sequence ϕ will be called a dictionary. It enables the sparse presentation for a signal.

One of the useful research orientation is called the online dictionary learning [22,23] which can speed up the convergence and improve the convergence result. There have been many different learning dictionary methods such as the MOD method, the Union of Orthobases method [24] and the Generalized PCA method [25].

3. **Our Proposed Method.** We use a smaller dimensional dictionary to learn and the reason is the easy decomposition for the smaller dictionary in the whole dictionary learning process:

Model

(1) Firstly, the original image is divided into blocks. Different tight wavelet frame transform is applied onto them.

(2) Secondly, to form a sparse model for a $\sqrt{n} \times \sqrt{n}$ image block as below.

$$\min ||\alpha_{i,j}||_0 \ s.t. \ X_{i,j} = \Phi\alpha_{i,j} \tag{19}$$

Here, α_i is the sparse representation for the image block and is the redundant dictionary.

(3) Thirdly, to form a large image sparse model, if all the blocks $X_{i,j}$ satisfies the following condition.

$$\alpha_{i,j} = \arg\min_{\alpha_{i,j}} ||\Phi\alpha_{i,j} - X_{i,j}||_2^2 + \mu||\alpha_{i,j}||_0 \tag{20}$$

And the whole sparse model can be written as follows.

$$\{\hat{\alpha}_{i,j}, \hat{X}\} = \arg\min_{\alpha_{i,j},X} \lambda||X - Y||_2^2 + \sum_{i,j} \mu_{i,j}||\alpha_{i,j}||_0 + \sum_{i,j} ||\Phi\alpha_{i,j} - R_{i,j}X||_2^2 \tag{21}$$

Here, $R_{i,j}$ is the extracting factor of the image blocks, i, j is the location for the blocks in the large image and λ is the similar factor.

Solve

(4) Fourthly, to solve the first decomposition model as follows.

$$\hat{\alpha}_{i,j} = \arg\min_{\alpha_{i,j},X} \sum_{i,j} \mu_{i,j}||\alpha_{i,j}||_0 + \sum_{i,j} ||\Phi\alpha_{i,j} - R_{i,j}X||_2^2 \tag{22}$$

The OMP method can be used to solve this problem that is to find the sparse representations of all the blocks which has the smallest distortion with the original image under the already known Φ. And we can find the sparse coefficients $\alpha_{i,j}$ under some special conditions for the distortion by stopping iterative computations.

(5) Fifthly, we will solve the following model.

$$\hat{X} = \arg\min_{\alpha_{i,j},X} \lambda||X - Y||_2^2 + \sum_{i,j} ||\Phi\alpha_{i,j} - R_{i,j}X||_2^2 \tag{23}$$

And the solution for the above has already known for us as follows.

$$\hat{X} = (\lambda I + \sum_{i,j} R_{i,j}^T R_{i,j})^{-1}(\lambda Y + \sum_{i,j} R_{i,j}^T \Phi\hat{\alpha}_{i,j}) \tag{24}$$

As we can see from the above analysis, all the solution methods are under the already known dictionary Φ. And here, we need to find the sparse dictionary through learning. So the problem (19) can be converted into the following one.

$$\{\hat{\Phi}, \hat{\alpha}_{i,j}\hat{X}\} = \arg\min_{\alpha_{i,j},X} \lambda||X - Y||_2^2 + \sum_{i,j} \mu_{i,j}||\alpha_{i,j}||_0 + \sum_{i,j} ||\Phi\alpha_{i,j} - R_{i,j}X||_2^2 \tag{25}$$

(6) Sixthly, divide the dictionary learning step and the sparse representation step by using the K-SVD method: (a) To find the image block sparse coefficients $\alpha_{i,j}$ under the well known redundant dictionary Φ . (b) To update every column of the dictionary and the sparse coefficients $\alpha_{i,j}$ by the representation of each image block.

(7) To initialize the parameters: let $X = Y$, Φ is the redundant tight wavelet frame dictionary. And the iterations number is J.

(8) To go through the following steps for J times:

- To get the solution for the following problem by applying the OMP method onto the image block $R_{i,j}X$. The solution is an approximate one and C, σ are constants.

$$\forall i,j \ \min_{\alpha_{i,j}} ||\alpha_{i,j}||_0 \ s.t. \ ||R_{i,j}X - \Phi\alpha_{i,j}||_2^2 \leq (C\sigma)^2 \tag{26}$$

- Update the dictionary
  - Find all the image blocks $R_{i,j}X$ which satisfy the condition $w_l = \{(i,j)|\alpha_{i,j}(l) \neq 0\}$.
  - Calculate the residual value for each coordinate $(i,j) \in w_l$:

$$e_{i,j}^l = R_{i,j}X - \sum_{m \neq l} d_m \alpha_{i,j}(m) \tag{27}$$

  - Let E_l be a residual matrix and the column vector is $\{e_{i,j}^l\}_{(i,j)\in w_l}$
  - Apply the Singular Value Decomposition (SVD) method onto E_l, then we can get $E_l = U\Delta V^T$. Let the first column of matrix U be the updated $\tilde{d}_l$ and at the same time multiply $\Delta(1,1)$ by the first column of $V$the result is used as the updated sparse coefficients.
- The final image can be shown below.

$$\hat{X} = (\lambda I + \sum_{i,j} R_{i,j}^T R_{i,j})^{-1}(\lambda Y + \sum_{i,j} R_{i,j}^T \Phi\hat{\alpha}_{i,j}) \tag{28}$$

One can solve the above problem by using the eigenvalue method.

4. **Experimental Results.** The 64 × 64 Lena image is used in our experiments here. The traditional DCT based CS method and ours are compared with each other in the following figure 1. All the results are gotten under the same reconstruction parameters.

Table 1. Comparison between the DCT based and wavelet frame based CS reconstruction results

L	DCT based CS	Wavelet basedf CS	Wavelet frame based CS
2000	20.33dB	20.99dB	22.98dB
2400	23.21dB	24.17dB	25.44dB
2800	25.99dB	26.60dB	27.35dB
3200	28.74dB	29.01dB	29.56dB
3600	31.27dB	32.38dB	33.89dB

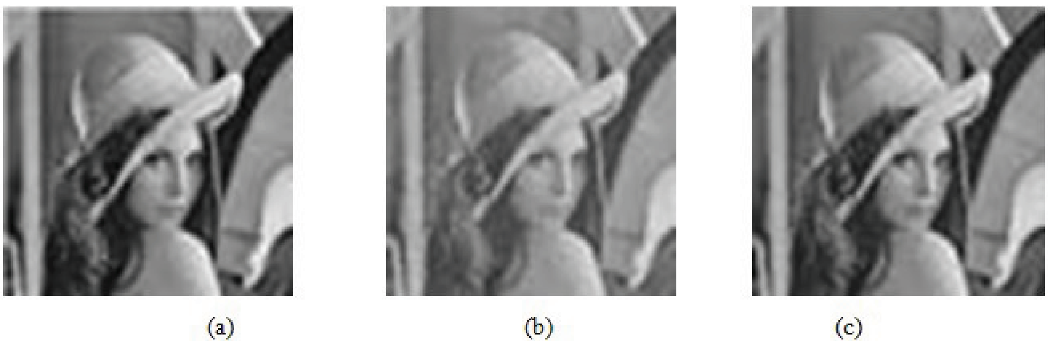

Figure 2. Reconstruction image comparison results: (a) original image, (b) DCT based CS reconstruction image, (c) wavelet frame based CS reconstruction image

As we can see from the above comparison results, our proposed method performs much better than the DCT based one. In fact, the method is also better than the traditional wavelet based one and the following table 1 gives the detailed data for the comparison and the parameter is the length of the observation sequence.

The PSNR values shown above gives a clear idea of the performance: our proposed method is much better than the traditional DCT and wavelet based CS methods.

5. **Conclusions.** A novel method for Compressed Sensing based on the wavelet frame and dictionary has been proposed in this paper. The algorithm design process and some experimental results have been shown. Good performance has shown the efficiency of our method. To decrease the computational complexity is our future work.

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